2007 Extension 2 Solution (by Terry Lee)

Q1

(a) $\int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\sin^{-1}\frac{2x}{3} + C.$

(b) $\int \tan^2 x \sec^2 x\, dx = \frac{\tan^3 x}{3} + C.$

(c) $\int_0^{\pi} x\cos x\, dx = [x\sin x]_0^{\pi} - \int_0^{\pi} \sin x\, dx$

$= [\cos x]_0^{\pi} = -2$

(d) $\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}}dx.$

Let $u^2 = 1-x, 2u\, du = -dx.$

When $x=0, u=1;$ when $x = \frac{3}{4}, u = \frac{1}{2}.$

$$\int_1^{\frac{1}{2}} \frac{1-u^2}{u}(-2udu) = -2\int_1^{\frac{1}{2}} \left(1-u^2\right)du$$

$$= 2\left[u - \frac{u^3}{3}\right]_{\frac{1}{2}}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}\right) - \left(\frac{1}{2}-\frac{1}{24}\right)\right]$$

$$= \frac{5}{12}.$$

(e) $\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}dx = \int_{\frac{1}{2}}^{2}\left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$$= \left[\ln(x+1) - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x\right]_{\frac{1}{2}}^{2}$$

$$= \ln\frac{3}{1.5} - \frac{1}{2}\ln\frac{5}{1.25} + \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

$$= \ln 2 - \frac{1}{2}\ln 4 + \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}2 - \tan^{-1}\frac{1}{2}$$

For discussion: This answer can be simplified further as $\tan^{-1}\frac{3}{4}$, but I wonder whether we must do it (since it has 4 marks)

Q2

(a) (i) $w = 4 - i$

(ii) $w - z = 4 - i - (4+i) = -2i$

(iii) $\frac{z}{w} = \frac{4+i}{4-i} = \frac{(4+i)^2}{(4-i)(4+i)} = \frac{15+8i}{17}.$

(b) (i) $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(ii) $(1+i)^{17} = \sqrt{2}^{17}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$

$$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 256\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= 256 + 256i.$$

(c) Let $z = x + iy.$

$$\frac{1}{z} + \frac{1}{\overline{z}} = \frac{1}{x+iy} + \frac{1}{x-iy}$$

$$= \frac{x-iy+x+iy}{x^2+y^2}$$

$$= \frac{2x}{x^2+y^2}$$

$$= 1$$

$\therefore 2x = x^2 + y^2.$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1.$

The locus is the circle of centre (1,0), radius 1, but excluding the origin as $z \neq 0.$

(d) (i) $OQ = OR$ rotates 60°, $\therefore z_2 = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \omega a.$

(ii) $OP = OR$ rotates (−60°),

$\therefore z_1 = a\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right) = a\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$\therefore z_1 z_2 = a^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$$= a^2\left(\cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}\right)$$

$$= a^2.$$

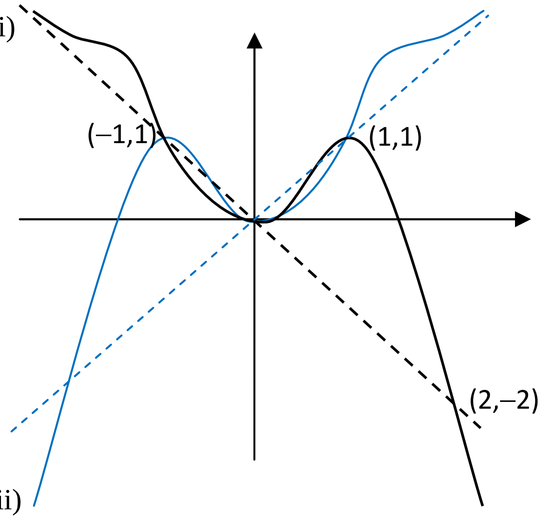
(iii) $z_1 + z_2 = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + a\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$$= 2a\cos\frac{\pi}{3} = a.$$

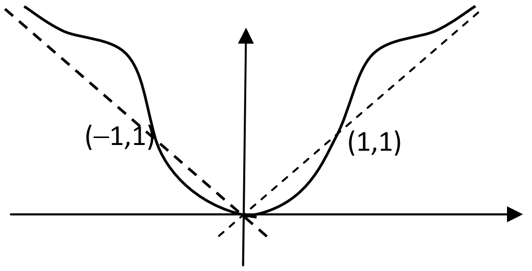
$\therefore z_1$ and z_2 are the roots of $z^2 - az + a^2$, since $\sum\alpha = a$, and $\prod\alpha = a^2.$

Q3

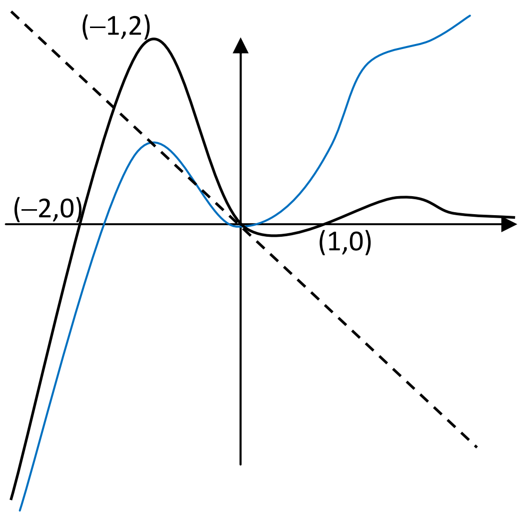
(a) (i)



(ii)



(iii) It's the same as adding $f(x)$ with $-x$.



(b) Let $y = 2x, \therefore x = \dfrac{y}{2}$

Substituting to the equation,

$$\frac{y^3}{8} - \frac{5y}{2} + 3 = 0$$

$y^3 - 20y + 24 = 0.$

$\therefore$ The required cubic is $y^3 - 20y + 24$ or $x^3 - 20x + 24$.

(c) $\delta V = 2\pi xy\delta x.$

$$V = 2\pi \int_1^e xy\,dx$$
$$= 2\pi \int_1^e x\frac{\ln x}{x}\,dx$$
$$= 2\pi \int_1^e \ln x\,dx$$
$$= 2\pi\left[x\ln x - x\right]_1^e$$
$$= 2\pi\left((e-e)-(0-1)\right)$$

$= 2\pi$ u^3.

(d) (i) Horizontally, $F\cos\theta - N\sin\theta = mr\omega^2$

$\therefore N\sin\theta = F\cos\theta - mr\omega^2$ (1)

Vertically, $F\sin\theta + N\cos\theta = mg$

$\therefore N\cos\theta = mg - F\sin\theta$ (2)

$(1)\times\sin\theta + (2)\times\cos\theta$ gives

$$N = F\cos\theta\sin\theta - mr\omega^2\sin\theta + mg\cos\theta - F\sin\theta\cos\theta$$
$$= mg\cos\theta - mr\omega^2\sin\theta$$

(ii) $N > 0$ then $mg\cos\theta > mr\omega^2\sin\theta$

$$\therefore \omega^2 < \frac{g\cos\theta}{r\sin\theta}$$
$$= \frac{g}{r}\cot\theta.$$

Q4

(a) $\angle LAM = \angle LAP + \angle PAB + \angle BAM$

But $\angle LAP = \angle LBP$ (angles subtending the same arc are equal)

$\angle LBP = \angle MBQ$ (vertically opposite angles)

$\angle MBQ = \angle MAQ$ (angles subtending the same arc are equal)

$\therefore \angle LAM = \angle PAB + \angle BAM + \angle MAQ$

$= \angle PAQ.$

(b) (i) $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta.$

(ii) $3\sin\theta\cos^2\theta - \sin^3\theta$

$$= \sin\theta\left(3\cos^2\theta - \sin^2\theta\right)$$

$$= \sin\theta\left(\sqrt{3}\cos\theta + \sin\theta\right)\left(\sqrt{3}\cos\theta - \sin\theta\right)$$

$$= 4\sin\theta\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$$

$$= 4\sin\theta\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)\left(-\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)$$

$$= 4\sin\theta\left(\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right)$$
$$\times\left(\sin\theta\cos\frac{2\pi}{3} + \cos\theta\sin\frac{2\pi}{3}\right)$$

$$= 4\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right).$$

(iii) $\sin\theta\sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta + \frac{2\pi}{3}\right) = \frac{1}{4}\sin 3\theta.$

$\therefore$ Its maximum value is $\frac{1}{4}$.

(c) The slice $PQRS$ has side length $e - x$.

Its area $= (e-x)^2$.

$\delta V = (e-x)^2\,\delta y$

$= \left(e - e^y\right)^2\delta y$

$= \left(e^2 - 2e^{y+1} + e^{2y}\right)\delta y$

$$\therefore V = \int_0^1 \left(e^2 - 2e^{y+1} + e^{2y}\right)dy$$

$$= \left[e^2 y - 2e^{y+1} + \frac{1}{2}e^{2y}\right]_0^1$$

$$= \left(e^2 - 2e^2 + \frac{1}{2}e^2\right) - \left(-2e + \frac{1}{2}\right)$$

$$= -\frac{1}{2}e^2 + 2e - \frac{1}{2}$$

$$= \frac{-e^2 + 4e - 1}{2}\ \text{units}^3.$$

(d) (i) $\sum\alpha = \alpha - \alpha + \beta = \beta = -q.$

Substituting x by $-q$ gives $-q^3 + q^3 - qr + s = 0$.

$\therefore qr = s.$

(ii) By inspection, $x^3 + qx^2 + rx + s = (x+q)\left(x^2 + \frac{s}{q}\right)$

$\therefore x = -q$ or $x^2 = -\frac{s}{q}$, i.e. $x = \pm i\sqrt{\frac{s}{q}}$, $\therefore$ There are two imaginary roots.

Q5

(a) (i) $\frac{{}^{12}C_3 \times {}^{12}C_3}{{}^{24}C_6} = 0.36$

(ii) $\frac{{}^{12}C_4 \times {}^{12}C_2 + {}^{12}C_5 \times {}^{12}C_1 + {}^{12}C_6}{{}^{24}C_6} = 0.32.$

Alternatively, $\frac{1}{2} - \frac{1}{2} \times 0.36 = 0.32.$

(b) (i) Differentiating both sides of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$.

$\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$.

$\therefore$ The gradient to the curve at $P(x_1, y_1)$ is $\frac{b^2 x_1}{a^2 y_1}$.

The equation of the tangent:

$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$.

$a^2 y_1 y - a^2 y_1^{\,2} = b^2 x_1 x - b^2 x_1^{\,2}$.

$b^2 x_1 x - a^2 y_1 y = b^2 x_1^{\,2} - a^2 y_1^{\,2}$.

$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = \frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2}$, on dividing both sides by $a^2 b^2$.

$\therefore \frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$, since (x_1, y_1) belongs to the ellipse.

(ii) $T(x_0, y_0)$ belongs to both TP and TQ, $\therefore$ its coordinates must satisfy both the tangents at P and Q, i.e.

$\frac{x_1 x_0}{a^2} - \frac{y_1 y_0}{b^2} = 1$ and $\frac{x_2 x_0}{a^2} - \frac{y_2 y_0}{b^2} = 1$.

$\therefore$ The equation of PQ must be $\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$, as (x,y) can be replaced by (x_1, y_1) or (x_2, y_2) .

(iii) Substituting (x, y) by $(ae, 0)$

$\frac{a e x_0}{a^2} - 0 = 1$

$\therefore x_0 = \frac{a}{e}$.

$\therefore T$ lies in the directrix.

(c) (i) $(x-1)(5-x) = -5 + 6x - x^2$

$= 4 - (x-3)^2$.

$\therefore a = 3, b = 2.$

(ii) $\int_1^5 \sqrt{(x-1)(5-x)}\,dx = \int_1^5 \sqrt{4-(x-3)^2}\,dx.$

Let $x - 3 = 2\sin\theta, dx = 2\cos\theta\, d\theta.$

When $x = 1, \theta = -\frac{\pi}{2}$; When $x = 5, \theta = \frac{\pi}{2}.$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$

$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2\theta + 1)\, d\theta$

$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 2\pi.$

(d) (i) $AC = 2AF = 2\cos\frac{\pi}{5}$. Similarly, $AD = 2\cos\frac{\pi}{5}$.

The angle sum of a polygon $= (n-2)\pi, \therefore$ the angle sum of the pentagon = 3π.

$\therefore \angle BAE = \frac{3\pi}{5}.$

$\therefore \angle CAD = \frac{\pi}{5}.$

By the Cosine rule,

$CD^2 = AC^2 + AD^2 - 2AC.AD.\cos\frac{\pi}{5}$

$1 = 4\cos^2\frac{\pi}{5} + 4\cos^2\frac{\pi}{5} - 2\times 4\cos^2\frac{\pi}{5}\cos\frac{\pi}{5}$

$1 = 8u^2 - 8u^3$, given $u = \cos\frac{\pi}{5}.$

$\therefore 8u^3 - 8u^2 + 1 = 0.$

(ii) $8u^3 - 8u^2 + 1 = (2u-1)(4u^2 - 2u - 1).$

Solving $4u^2 - 2u - 1 = 0$ gives $u = \frac{1 \pm \sqrt{5}}{4}.$

$\therefore \cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$, since $\cos\frac{\pi}{5} > 0.$

Q6

(a) (i) Let $a = b = 1$,

$$2^n = 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

$$\therefore 2^n > \binom{n}{2}, \text{ for all } n \geq 2.$$

(ii) From (i) above, $2^n > \dfrac{n(n-1)}{2}$

$$\frac{1}{2^n} < \frac{2}{n(n-1)}$$

$$\frac{1}{2^{n-1}} < \frac{4}{n(n-1)}$$

$$\frac{n+2}{2^{n-1}} < \frac{4(n+2)}{n(n-1)}.$$

(iii) Let $n = 1$, LHS $= 1$, RHS $= 4 - 3 = 1$, $\therefore$ true!

Assume the statement is true for $n = k$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$$

$$\therefore \text{RTP } 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots + (k+1)\left(\frac{1}{2}\right)^k = 4 - \frac{k+3}{2^k}$$

$$\text{LHS} = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$$

$$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$$

$$= 4 - \frac{k+2}{2^{k-1}} - \frac{-k-1}{2^k}$$

$$= 4 - \frac{2(k+2) - k - 1}{2^k}$$

$$= 4 - \frac{k+3}{2^k}$$

$$= \text{RHS}.$$

$\therefore$ It's true for $n = k + 1$.

$\therefore$ It's true for $n \geq 1$.

(iv) From (ii), $\dfrac{n+2}{2^{n-1}} < \dfrac{4(n+2)}{n(n-1)}$.

As $n \to \infty$, $\dfrac{4(n+2)}{n(n-1)} \to 0^+$, $\therefore \dfrac{n+2}{2^{n-1}} \to 0^+$

$\therefore$ The limiting sum of $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots$ is 4.

(b) (i) $v = \dfrac{dx}{dt} = 5\dfrac{1.4e^{1.4t} - 1.4e^{-1.4t}}{e^{1.4t} + e^{-1.4t}}$

$$= 7\frac{e^{1.4t} - e^{-1.4t}}{e^{1.4t} + e^{-1.4t}}.$$

(ii) $v^2 = 49\left(\dfrac{e^{1.4t} - e^{-1.4t}}{e^{1.4t} + e^{-1.4t}}\right)^2$

$$= 49\left(\frac{e^{2.8t} + e^{-2.8t} - 2}{e^{2.8t} + e^{-2.8t} + 2}\right)$$

$$= 49\left(\frac{e^{2.8t} + e^{-2.8t} + 2 - 4}{e^{2.8t} + e^{-2.8t} + 2}\right)$$

$$= 49\left(1 - \left(\frac{2}{e^{1.4t} + e^{-1.4t}}\right)^2\right)$$

But $\dfrac{x}{5} = \ln\dfrac{e^{1.4t} + e^{-1.4t}}{2}$, $\therefore \dfrac{e^{1.4t} + e^{-1.4t}}{2} = e^{\frac{x}{5}}$,

$$\therefore \left(\frac{2}{e^{1.4t} + e^{-1.4t}}\right)^2 = e^{-\frac{2x}{5}}.$$

$$\therefore v^2 = 49\left(1 - e^{-\frac{2x}{5}}\right)$$

(iii) $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{49}{2}\dfrac{d}{dx}\left(1 - e^{-\frac{2x}{5}}\right)$

$$= \frac{49}{2} \times \frac{2}{5}e^{-\frac{2x}{5}}$$

$$= \frac{49}{5}e^{-\frac{2x}{5}}$$

$$= \frac{49}{5}\left(1 - \frac{v^2}{49}\right)$$

$$= 9.8 - 0.2v^2.$$

(iv) $-0.2v^2$ is the air resistance.

As its velocity increases from 0 to its terminal velocity u its acceleration decreases from 9.8 to zero.

(v) Its terminal velocity is $v = \sqrt{\dfrac{9.8}{0.2}} = 7$ m/s.

Q7

(a) (i) Let $g(x) = \sin x - x$

$g'(x) = \cos x - 1 < 0$ for $x > 0, \therefore g(x)$ is decreasing for all $x > 0$.

When $x = 0, g(0) = 0$,

$\therefore g(x) < 0$

$\therefore \sin x - x < 0$

$\therefore \sin x < x$ for $x > 0$.

(ii) $f(x) = \sin x - x + \dfrac{x^3}{6}$.

$f'(x) = \cos x - 1 + \dfrac{x^2}{2}$.

$f''(x) = -\sin x + x$.

From (i), $f''(x) > 0, \therefore f(x)$ is concave up.

(iii) As $f''(x) > 0$, $f'(x)$ is increasing. When $x = 0$, $f'(0) = 0$,

$\therefore f'(x) > 0$.

As $f(x)$ is concave up and its gradient is positive, $f(x) > 0$.

$\therefore \sin x - x + \dfrac{x^3}{6} > 0$.

$\therefore \sin x > x - \dfrac{x^3}{6}$.

(b) (i) In ΔPUR and ΔQVR,

$\angle R$ is common,

$\angle PUR = \angle QVR = 90°$,

$\therefore \Delta PUR \,|||\, \Delta QVR$.

$\therefore \dfrac{PR}{QR} = \dfrac{PU}{QV}$ (corresponding sides in similar triangles are proportional).

(ii) But $PS = ePU$, and $QS = eQV$,

$\dfrac{PU}{QV} = \dfrac{ePU}{eQV} = \dfrac{PS}{QS}$.

(iii) In $\Delta PRS, \dfrac{\sin(\phi + \theta)}{\sin \alpha} = \dfrac{PR}{PS}$. (1)

In $\Delta QRS, \dfrac{\sin \theta}{\sin \alpha} = \dfrac{QR}{QS}$. (2)

$\dfrac{(1)}{(2)}$ gives $\dfrac{\sin(\phi + \theta)}{\sin \theta} = \dfrac{PR}{PS} \times \dfrac{QS}{QR}$

$= \dfrac{PR}{QR} \times \dfrac{QS}{PS}$

$= 1$, from (i) and (ii).

$\therefore \sin(\phi + \theta) = \sin \theta$

$\therefore \phi + \theta = \theta$ or $\pi - \theta$

$\therefore \phi = \pi - 2\theta$, since $\phi \neq 0$.

(iv) As $\phi \to 0, \pi - 2\theta \to 0, \therefore \theta \to \dfrac{\pi}{2}$.

(c) (i) Since $PS = ePN, \dfrac{PS}{PR} = e\dfrac{PN}{PR} = e \cos \beta$.

(ii) Similarly, $\dfrac{PS'}{PW} = e\dfrac{PM}{PW} = e \cos \beta$.

$\therefore \dfrac{PS}{PR} = \dfrac{PS'}{PW}$.

In $\Delta PRS, \dfrac{PS}{PR} = \cos(\angle RPS)$, in $\Delta PWS', \dfrac{PS'}{PW} = \cos(\angle WPS')$,

$\therefore \angle RPS = \angle WPS'$.

Q8

(a) (i) Let $u = a - x, du = -dx.$

When $x = 0, u = a$; When $x = a, u = 0.$

$$\int_0^a f(x)dx = \int_a^0 f(a-u)\,(-du)$$
$$= \int_0^a f(a-u)\,du$$
$$= \int_0^a f(a-x)\,dx.$$

(ii) $$\int_0^a f(x)dx = \frac{1}{2}\left(\int_0^a f(x)dx + \int_0^a f(a-x)dx\right)$$
$$= \frac{1}{2}\int_0^a \left(f(x) + f(a-x)\right)dx$$
$$= \frac{1}{2}\int_0^a f(a)dx$$
$$= \frac{1}{2} f(a)\int_0^a dx$$
$$= \frac{1}{2} f(a)\times\left[x\right]_0^a$$
$$= \frac{a}{2} f(a).$$

(b) (i) This is a GP, with $a = 1, r = z^2$.

$$S_n = \frac{z^{2n}-1}{z^2-1} = z^{n-1}\frac{z^{n+1}-\frac{1}{z^{n-1}}}{z^2-1}$$
$$= z^{n-1}\frac{\left(z^n - \frac{1}{z^n}\right)z}{\left(z - \frac{1}{z}\right)z}$$
$$= \frac{z^n - z^{-n}}{z - z^{-1}} z^{n-1}.$$

(ii) Let $z = \cos\theta + i\sin\theta$, by De Moivre's theorem,

$z^n = \cos n\theta + i\sin n\theta.$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$, as $\cos x$ is even, and $\sin x$ is odd.

$$\therefore \text{LHS} = 1 + \cos 2\theta + ... + \cos(2n-2)\theta$$
$$+ i\left(\sin 2\theta + ... + \sin(2n-2)\theta\right).$$

$$\text{RHS} = \frac{\cos n\theta + i\sin n\theta - \left(\cos n\theta - i\sin n\theta\right)}{\cos\theta + i\sin\theta - \left(\cos\theta - i\sin\theta\right)} \times\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]$$
$$= \frac{2i\sin n\theta}{2i\sin\theta}\times\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]$$
$$= \frac{\sin n\theta}{\sin\theta}\times\left[\cos(n-1)\theta + i\sin(n-1)\theta\right].$$

(iii) Let $\theta = \frac{\pi}{2n}$, by considering the imaginary parts in part (ii),

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} ... + \sin\frac{(n-1)\pi}{n} = \frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}\sin\frac{(n-1)\pi}{2n}$$
$$= \frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}\sin\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) = \frac{1}{\sin\frac{\pi}{2n}}\cos\frac{\pi}{2n} = \cot\frac{\pi}{2n}.$$

(c) (i) $$d_1 + d_2 + ... + d_{n-1} = \frac{\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + ... + \sin\frac{(n-1)\pi}{n}}{\sin\frac{\pi}{n}}$$
$$= \frac{\cot\frac{\pi}{2n}}{\sin\frac{\pi}{n}} = \frac{\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{1}{2\sin^2\frac{\pi}{2n}}.$$

(ii) $$\frac{p}{q} = \frac{n}{\frac{1}{n}\times\frac{1}{2\sin^2\frac{\pi}{2n}}} = 2n^2\sin^2\frac{\pi}{2n}.$$

(iii) $$\frac{p}{q} = \frac{\pi^2}{2}\frac{\sin^2\frac{\pi}{2n}}{\frac{\pi^2}{4n^2}}.\text{As } n\to\infty, \frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\to 1, \frac{p}{q}\to\frac{\pi^2}{2}.$$